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ON f-VECTORS AND BETTI NUMBERS OF MULTICOMPLEXES

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A multicomplex M is a collection of monomials closed under divisibility. For such M we construct a cell complex Δ_M whose i-dimensional cells are in bijection with the f_i monomials of M of degree $i+1$. The bijection is such that the inclusion relation of cells corresponds to divisibility of monomials. We then study relations between the numbers f_i and the Betti numbers of Δ_M. For squarefree monomials the construction specializes to the standard geometric realization of a simplicial complex.

1. Introduction

The well known Kruskal-Katona theorem (see [5] and [6]) characterizes the f-vectors of simplicial complexes. It states that some ultimately vanishing sequence $f=(f_0,f_1,\ldots)$ of non-negative integers is the f-vector of some simplicial complex if and only if the following system of inequalities is satisfied:

$$\partial_k(f_k) \leq f_{k-1},\ k \geq 1.$$

Here, the number-theoretic function $\partial_k(n)$ is defined in the following way. For $n,k \geq 1$ the integer n can be uniquely expressed in the following form

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_i}{i},$$

where $a_k > a_{k-1} > \ldots > a_i \geq i \geq 1$. Then we define:

$$\partial_{k-1}(n) = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \ldots + \binom{a_i}{i-1},$$

and we also let $\partial_{k-1}(0)=0$.

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To each simplicial complex we can associate in a natural way a family of square-free monomials in the polynomial ring over the vertices of this complex as the indeterminates, so that some monomial belongs to the family if and only if its vertices span a simplex of the corresponding simplicial complex. More generally, we say that a collection M of monomials in the polynomial ring $\mathbf{R}[x_1,\ldots,x_n]$ form a *multicomplex* if it is closed under divisibility, i.e. if $u\in M$ and $v\mid u$ imply $v\in M$. Therefore the Kruskal-Katona theorem can be regarded as a result about f-vectors of multicomplexes of square-free monomials in the corresponding polynomial ring. So, it is natural to ask what can be said in the general case of multicomplexes (whose monomials are not necessarily all square-free).

Such a characterization exists since 1927 and is due to Macaulay (see [4, 6, 9]). It states that a sequence of non-negative integers is the f-vector of some multicomplex if and only if the following system of inequalities is satisfied:

$$\partial^k(f_k) \leq f_{k-1},\ k \geq 1.$$

Here we call $f=(f_0,f_1,\ldots)$ the *f-vector* of a (not necessarily finite) multicomplex M if f_i is the number of monomials of degree $i+1$ for all $i\geq 0$, and given the k-binomial expansion of n as above we define

$$\partial^{k-1}(n) = \binom{a_k-1}{k-1} + \binom{a_{k-1}-1}{k-2} + \ldots + \binom{a_i-1}{i-1},$$

and also $\partial^{k-1}(0)=0$.

Let us now denote with β_k the k-th reduced Betti number of a simplicial complex over some field. The β-vector of the simplicial complex is the (ultimately vanishing) sequence $\beta=(\beta_0,\beta_1,\ldots)$ of its Betti numbers. In [2] a characterization of f-vectors and β-vectors (or (f,β)-pairs, for short) of simplicial complexes is obtained. This characterization states that two ultimately vanishing sequences of non-negative integers are f-vector and β-vector of some simplicial complex if and only if $\chi_{-1}=1$ and

$$\partial_k(\chi_k+\beta_k) \leq \chi_{k-1},\ k \geq 1,$$

where $\chi_{k-1}=\sum_{j\geq k}(-1)^{j-k}(f_j-\beta_j)$ for $k\geq 0$.

The purpose of this paper is to seek a multicomplex analogue of this result, in the same way that Macaulay's theorem is an analogue of the Kruskal-Katona theorem. For this question even to make sense we must have a reasonable notion of Betti numbers for multicomplexes. What we have in mind demands a CW complex whose i-dimensional cells correspond bijectively to the degree $i+1$ monomials of the multicomplex in such a way that the inclusion of closed cells corresponds to divisibility of monomials. In Section 2 we will construct such cell complexes. Their construction is essentially combinatorial (the attaching maps are completely determined by the underlying poset of monomials), and they specialize to simplicial complexes in the square-free case. We take the Betti numbers of such CW complexes as the β-sequences of multicomplexes. These CW complexes seem very natural and may be of interest also for other purposes.

In Section 3 we begin the study of (f,β)-pairs of multicomplexes. The original goal was to achieve a complete characterization, but this has not yet been reached. As partial progress we prove one sufficient condition (rather easy) and one necessary condition (rather more difficult).

The final section discusses some alternative approaches.

2. Cellular realization of a multicomplex

The homology we want to associate to a multicomplex has a simple combinatorial description entirely in terms of the monomials themselves. We begin with this and will then continue with a description of the underlying cell complex.

For a general multicomplex M we will use the homology theory provided via the boundary operator defined by

$$d\left(x_{i_0}^{\alpha_0}\dots x_{i_k}^{\alpha_k}\right)=\sum_{j=0}^{k}(-1)^{\alpha_0+\dots+\alpha_{j-1}}\cdot r_j\cdot x_{i_0}^{\alpha_0}\dots x_{i_j}^{\alpha_j-1}\dots x_{i_k}^{\alpha_k},$$

where $r_j=0$ if α_j is even and $r_j=1$ if α_j is odd. This definition agrees with the definition of the boundary operator in the simplicial category applied to the monomial expressed in the following form:

$$x_{i_0}^{\alpha_0}\dots x_{i_k}^{\alpha_k}=\underbrace{x_{i_0}\dots x_{i_0}}_{\alpha_0}\dots\underbrace{x_{i_k}\dots x_{i_k}}_{\alpha_k}.$$

Therefore it is easy to see that $d^2=0$, and so we have a boundary operator.

Of course, each monomial can be uniquely represented in the form of the product of a full square monomial and a squarefree monomial $m=p^2q$, for some $p,q\in M$. Let us denote

$$M_{p^2}=\{q\in M\mid p^2q\in M,\ q\text{ squarefree}\}.$$

Each of these M_{p^2} is easily seen to be a simplicial complex and so, a multicomplex can be represented as a union of some simplicial complexes over its full square monomials. Each k-dimensional simplex of M_{p^2} contributes to $f_{k+2|p|}$.

Directly from the definition of the boundary operator it follows that the boundary of some monomial is completely contained in the simplicial complex over the same full square. Namely, one easily verifies the relation

$$d(p^2q)=p^2dq,$$

where p and q are monomials of this multicomplex. So, the homology of the multicomplex also splits (as the f-vector does) into a direct sum of the homology groups of the above mentioned simplicial complexes, and we have:

$$H_*(M)\cong\bigoplus_{p^2\in M}H_{*-2|p|}(M_{p^2}).$$

From these considerations we can easily derive the following computations for some special multicomplexes. Let us consider any simplicial complex K on the vertex set $V = \{x_1, \ldots, x_n\}$ and let us associate to it a multicomplex in the polynomial ring over the indeterminates $\{x_1, \ldots, x_n\}$ described by

$$M_K = \{m \mid \mathrm{supp}(m) \in K\}.$$

These monomials are the additive generators of the Stanley-Reisner ring of K. For this multicomplex one easily verifies the following.

Proposition 2.1. $H_*(M_K) \cong H_*(K)$.

Proof. For any $m \in M_K$ let us denote $\mathrm{supp}(m) = \sigma$. Now we easily see that for the geometric realization of the simplicial complex $(M_K)_{m^2}$ we have

$$|(M_K)_{m^2}| \approx \sigma * \mathrm{Lk}_K(\sigma).$$

Since this complex is contractible for each $m \in M_K \setminus \{1\}$ we see that the only simplicial complex that contributes to the homology of M_K is the one of squarefree monomials (over 1), which coincides with K, and the result is established. ∎

By the same considerations we obtain for the multicomplex $M_{K,r}$ of monomials from M_K with all exponents not greater than r that:

$$H_*(M_{K,r}) \cong \begin{cases} H_*(K), & r \quad \text{odd}, \\ \oplus_{\sigma \in K} H_{*-r|\sigma|}(\mathrm{Lk}_K(\sigma)), & r \ \text{even}. \end{cases}$$

According to K. S. Sarkaria (private communication) this result on $M_{K,r}$ has also been obtained by T. Bier in another context.

We will now describe a topological model for general multicomplexes playing the same role as geometric simplicial complexes do for the special multicomplexes of square-free monomials. Having in mind the given definition of the boundary operator the following will seem natural.

To the monomial x^{k+1} we associate the quotient space obtained from the k-dimensional simplex $\sigma^k = (a_0, \ldots, a_k)$ by identifying all of its $(k-1)$-dimensional faces (with the induced order of vertices). Let us denote this quotient space by τ^k. Of course, τ^0 is a point, τ^1 is the circle, τ^2 is the dunce hat, and so on. The spaces τ^k are studied in [1], where they are called higher-dimensional dunce hats and where it is shown that they are contractible for even k (but not collapsible if $k > 0$), and are homotopy k-spheres for odd k.

To the monomial $x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}$ we associate the join of the corresponding quotient spaces, i.e. the space

$$\tau(x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}) = \tau_1^{\alpha_1 - 1} * \ldots * \tau_k^{\alpha_k - 1}.$$

Of course, this space could also be obtained from the $(\alpha_1 + \ldots + \alpha_k - 1)$-dimensional simplex by the appropriate identifications.

Proposition 2.2. *The cell $\tau(m)$ associated with a monomial m has the following homotopy type:*

$$\tau(m) \simeq \begin{cases} S^{\alpha-1}, & \text{if } m \text{ is a full square of degree } \alpha \\ \text{point}, & \text{otherwise.} \end{cases}$$

Proof. For $m = x^\alpha$ this is proved in [1], as was already mentioned. The general fact follows via the following standard properties of the join construction:

(i) $A \simeq B$ and $A' \simeq B'$ imply $A * A' \simeq B * B'$

(ii) if A is contractible then $A * B$ is contractible

(iii) $S^k * S^\ell \cong S^{k+\ell+1}$. ∎

Now, let M be a multicomplex. To each monomial $m \in M$ we associate a cell $\tau(m)$ of the above type, and then we glue them together by the obvious characteristic maps to form a CW complex Δ_M. It is easy to see from the construction that the boundary operator of the cellular chain complex of Δ_M corresponds to the boundary operator of our homology theory for multicomplexes. So, Δ_M has the same homology as that given by the the combinatorial boundary operator described above, and hence it yields the β-vector of our multicomplex.

Let us point out that in the case of a multicomplex of squarefree monomials this construction reduces to the usual geometric realization of simplicial complexes.

3. On (f,β)-pairs of multicomplexes

Recall the definitions of ∂^k and χ_k given in the introduction. By analogy with the case of simplicial complexes [2] and certain cell complexes [3], one could suspect that the relations

$$\partial^k(\chi_k + \beta_k) \leq \chi_{k-1},\ k \geq 1,$$

might characterize (f,β)-pairs of multicomplexes. However, many examples of multicomplexes not satisfying some of these relation can be found. For example, the multicomplex $M = \{1, x_1, \ldots, x_n, x_1^2, \ldots, x_n^2\}$ has $f = (n,n)$, $\beta = (n-1,n)$ and $\partial^1(\chi_1 + \beta_1) = \partial^1(n) > 0 = \chi_0$. Still, using a proof analogous to that in [2] we see that these relations give a sufficient condition for the existence of a multicomplex with given (f,β)-pair.

Theorem 3.1. *Given two sequences $f = (f_0, f_1, \ldots)$ and $\beta = (\beta_0, \beta_1, \ldots)$ of nonnegative integers there is a multicomplex having these f- and β-vectors if the relations $\chi_{-1} = 1$ and*

$$\partial^k(\chi_k + \beta_k) \leq \chi_{k-1},\ k \geq 1$$

are satisfied.

Proof. Let us take the first χ_k monomials of degree $k+1$ (or of dimension k) in reverse lexicographic order and then the next β_k of them, for all k.

By Macaulay's theorem and the given relations, these monomials form a multicomplex M on indeterminates $x_1, x_2, \ldots$ for which the shadow of the k-dimensional monomials is contained among the first χ_{k-1} monomials of dimension $k-1$. Now, let us add to M the new indeterminate x_0 and the products of x_0 with the first χ_k monomials of dimension k, for all k. It is easy to check that in this way we obtain a multicomplex $M'$with the given f-vector.

The homology of M' splits into the homology groups of simplicial complexes over the full square monomials. But, in each of these simplicial complexes, we obtain cones with the apex at x_0 and then some additional monomials not extendable by x_0 (so called "near-cones", see [2]). There are altogether β_k of these additional monomials of dimension k, and since it is easy to see that all these simplicial complexes are shifted (see [2]), the constructed multicomplex has the required homology groups. ∎

The relations $\chi_k \geq 0$ have to be satisfied for $k \geq 1$, since it is shown in [2] that CW complexes satisfy these relations. But, these relations are not sufficient condition for (f, β)-pairs of multicomplexes, since the CW complexes constructed above are of a special type (e.g. the intersection of two closed cells is a closed cell). For instance, for $f = (n, n, n)$, $\beta = (n-1, n, n)$ the relations $\chi_k \geq 0$ are satisfied, but there is no multicomplex with such an (f, β)-pair.

Another necessary relation that is easy to prove is

$$\partial^1(f_1 - \beta_0) \leq f_0 - \beta_0.$$

In order to establish one non-trivial necessary condition for (f, β)-pairs of multicomplexes, we need the following property of the number-theoretic function ∂^k. A similar property of the function ∂_k was proved by Eckhoff and Wegner in [5].

Lemma 3.2. *The functions ∂^k $(k \geq 1)$ satisfy the relations:*

$$\partial^k(m_0 + m_1 + \ldots + m_k + 1) \leq \sum_{i=0}^{k-1} \max\left\{\partial^{k-i}(m_i), m_{i+1}\right\} + 1.$$

Proof. On the set of indeterminates $\{x_0, x_1, \ldots\}$ let us consider the set of monomials of degree $k+1$ which are divisible by x_0^i but not by x_0^{i+1}. Let us denote by A_i the set of the m_i first of them in reverse lexicographic order, for each $i = 0, 1, \ldots, k$, and let $A_{k+1} = \left\{x_0^{k+1}\right\}$.

Furthermore, let $S = \mathrm{Shadow}(A_0 \cup \ldots \cup A_{k+1})$, the set of all monomials of degree k which divide some of the monomials from the union $A_0 \cup \ldots \cup A_{k+1}$. The monomials in S divisible by x_0^i but not by x_0^{i+1} can be obtained by dividing some monomial from A_i by an indeterminate different from x_0, or by dividing some monomial from A_{i+1} by x_0.

Since the monomials in each family A_i are compressed (after dividing them by the corresponding power of x_0), there are $\max\left\{\partial^{k-i}(m_i), m_{i+1}\right\}$ monomials in S with x_0^i as the exact power of x_0, for $i=0, 1, \ldots, k-1$. Of course, there is exactly one monomial in S divisible by x_0^k. Therefore,

$$|S| = \sum_{i=0}^{k-1} \max\left\{\partial^{k-i}(m_i), m_{i+1}\right\} + 1.$$

On the other hand, using Macaulay's theorem we get

$$\partial^k(m_0 + m_1 + \ldots + m_k + 1) = \partial^k(|A_0 \cup \ldots \cup A_{k+1}|) \leq |S|,$$

establishing in this way the lemma. ∎

Remark. If $m_k = 0$, we can take $A_k = A_{k+1} = \emptyset$, and with the obvious modifications we get for $k \geq 1$ the relation

$$\partial^k(m_0 + m_1 + \ldots + m_{k-1}) \leq \sum_{i=0}^{k-1} \max\left\{\partial^{k-i}(m_i), m_{i+1}\right\}.$$

Theorem 3.3. *The f-vector and the Betti numbers of a multicomplex satisfy the following relations for $k \geq 1$:*

$$\partial^k(f_{2k+1} + \beta_{2k}) \leq f_{2k-1} - \beta_{2k-1}.$$

Proof. We give an inductive argument (on the size of the multicomplex) using the previous lemma. The indeterminates of our multicomplex M will be $x_0, x_1, \ldots$.

Part 1: Assume first that not all monomials in M are square-free. Without loss of generality we assume that $x_0^2 \in M$. As was already said, the f-vector and the homology groups of the multicomplex split into those of the simplicial complexes over full square monomials. We will denote by f_j^i and β_j^i the number of j-dimensional faces and the number of generators of the j-th homology group of the multicomplex coming from the simplicial complexes over the full square monomials which are divisible by x_0^{2i} but are not divisible by x_0^{2i+2}. Then, of course, we have

$$\begin{aligned}\partial^k(f_{2k+1} + \beta_{2k}) &= \partial^k(f_{2k+1}^0 + \beta_{2k}^0 + \ldots + f_{2k+1}^k + \beta_{2k}^k + f_{2k+1}^{k+1}) \\ &\leq \sum_{i=0}^{k-1} \max\left\{\partial^{k-i}(f_{2k+1}^i + \beta_{2k}^i), f_{2k+1}^{i+1} + \beta_{2k}^{i+1}\right\} + 1.\end{aligned}$$

Here f_{2k+1}^{k+1} equals 0 or 1, and if it equals 0 and $f_{2k+1}^k = \beta_{2k}^k = 0$, then 1 on the right-hand side is not necessary by the previous remark.

Let us consider the monomials of the multicomplex divisible by x_0^{2i} and not by x_0^{2i+2}. If we divide all these monomials by x_0^{2i}, we obtain a multicomplex with f-vector $\overline{f}$, where $\overline{f}_{2(k-i)+1} = f^i_{2k+1}$, and similarly for Betti numbers. By the induction hypothesis we have for each $i=0, 1, \ldots, k-1$:

$$\partial^{k-i}(f^i_{2k+1} + \beta^i_{2k}) \leq f^i_{2k-1} - \beta^i_{2k-1}.$$

Claim. *For each* $i=0$, *1*, …, $k-1$, *we have*

$$f^{i+1}_{2k+1} + \beta^{i+1}_{2k} \leq f^i_{2k-1} - \beta^i_{2k-1}.$$

Proof of the claim. Let us consider the full square monomials of type $x_0^{2i}m^2$ and $x_0^{2i+2}m^2$ for some $m \in M$, and the simplicial complexes over them. In order to prove the statement of the claim it is enough to prove that for each such pair of simplicial complexes and their f-vectors and Betti numbers we have

$$f^{i+1}_{2k+1}(x_0^{2i+2}m^2) + \beta^{i+1}_{2k}(x_0^{2i+2}m^2) \leq f^i_{2k-1}(x_0^{2i}m^2) - \beta^i_{2k-1}(x_0^{2i}m^2).$$

Taking the monomials from these two simplicial complexes only and dividing them by $x_0^{2i}m^2$, we obtain a multicomplex which can be represented as the union of two simplicial complexes (one over 1 and the other over x_0^2). Let us denote this multicomplex with M'. Of course, it is enough to prove that this multicomplex satisfies (for every k) the inequality:

$$f_{2k+1}(x_0^2) + \beta_{2k}(x_0^2) \leq f_{2k-1}(1) - \beta_{2k-1}(1).$$

Let us consider the following simplicial complexes whose indeterminates will be x_1, x_2, … (i.e. all except x_0):

$$K_i = \{m \mid x_0 \nmid m, x_0^i \cdot m \in M'\},\ i = 0, 1, 2, 3.$$

Then obviously $K_3 \subseteq K_2 \subseteq K_1 \subseteq K_0$. Let us remark that the multicomplex M' can be built from the multicomplex

$$M'' = K_1 \cup \{x_0 \cdot m \mid m \in K_1\} \cup \{x_0^2 \cdot m \mid m \in K_2\} \cup \{x_0^3 \cdot m \mid m \in K_2\}$$

by successive application (in the appropriate order) of steps of the following kinds:

(a) removing one by one the monomials $x_0^3 \cdot m$, $m \in K_2 \setminus K_3$,

(b) adding one by one the monomials from $K_0 \setminus K_1$.

The multicomplexes M' and M'' consist of two simplicial complexes (now in the indeterminates x_0, x_1, …), one over 1 and one over x_0^2. In the case of the multicomplex M'', these simplicial complexes are cones with apex at x_0 over K_1 and K_2. So, these simplicial complexes are contractible, and the Betti numbers of M'' are all equal to 0. Since $K_2 \subseteq K_1$, we see that the multicomplex M'' satisfies the required inequality:

$$f_{2k+1}(x_0^2, M'') + \beta_{2k}(x_0^2, M'') \le f_{2k-1}(1, M'') - \beta_{2k-1}(1, M'').$$

When we apply a step of type (a) to a monomial of degree $2r+2$, we decrease $f_{2r+1}(x_0^2)$ by one and increase $\beta_{2r}(x_0^2)$ by one, so the left-hand side of the above inequality remains unchanged for $k = r$. For $k \neq r$ this is trivially true. When we apply a step of type (a) to a monomial of degree $2r+1$, all $f_{2k+1}(x_0^2)$ remain unchanged. Also, all $\beta_{2k}(x_0^2)$ remain the same except for $\beta_{2r}(x_0^2)$ which may remain the same or decrease by one. So, by applying steps of type (a) to some multicomplex we obtain a new multicomplex for which the left-hand side of the above inequalities either remains the same or decreases, and the right-hand side remains the same.

When we apply a step of type (b) to a monomial of degree $2r$, we increase $f_{2r-1}(1)$ by one and $\beta_{2r-1}(1)$ either increases by one or remains the same. So, the right-hand side of the above inequality for $k=r$ either remains the same or increases by one. For $k\neq r$ it trivially remains the same. If we apply a step of type (b) to a monomial of degree $2r+1$, all $f_{2k-1}(1)$ remain unchanged. Also, all $\beta_{2k-1}(1)$ remain the same except for $\beta_{2r-1}(1)$ which may remain the same or decrease by one. So by applying steps of type (b) to a multicomplex we obtain a new multicomplex for which the left-hand side remains unchanged and the right-hand side of the above inequality either remains the same or increases.

Since the considered inequalities are true for the multicomplex M'', and since M' can be obtained from M'' by applying steps of type (a) and (b), it follows that those inequalities hold also for the multicomplex M' (the left-hand side has decreased and the right-hand side increased). In this way the claim has been proved.

Using these two ingredients we establish the inductive step, since we have

$$\partial^k(f_{2k+1} + \beta_{2k}) \le \sum_{i=0}^{k-1}(f_{2k-1}^i - \beta_{2k-1}^i) + 1 = f_{2k-1} - \beta_{2k-1}.$$

The given argument provides a proof in the case $x_0^{2k} \in M$. In the other case $f_{2k+1}^k = \beta_{2k}^k = f_{2k+1}^{k+1} = 0$, and the reader will easily (using the above remark) modify the argument to establish that case as well.

Part 2: We have reduced the statement to the case of a simplicial complex. But in this case we can use algebraic shifting, see [2, 7]. We consider the shifted complex with the same f-vector and the same Betti numbers. Let's fix any $k \ge 1$.

For each generator of $2k$-dimensional homology (a maximal $2k$-simplex which cannot be extended by x_0) let us add a $(2k+1)$-dimensional simplex "killing" it (i.e. a cone over it with apex at x_0) and let us remove each maximal $(2k-1)$-dimensional simplex (without vertex x_0) generating $(2k-1)$-dimensional homology. In this way we obtain a simplicial complex having $f_{2k+1}+\beta_{2k}$ simplices of dimension $2k+1$ and $f_{2k-1}-\beta_{2k-1}$ simplices of dimension $2k-1$.

We finish our argument by showing inductively that for each simplicial complex

$$\partial^k(f_{2k+1}) \leq f_{2k-1}.$$

The argument is again based on induction on the size of the simplicial complex and very similar to the previous one. We decompose the simplicial complex (which we may assume to be compressed) into the simplicial complex of monomials not divisible by x_1x_2 (let us call it K_0), and the monomials divisible by x_1x_2. Let us divide these monomials by x_1x_2, and we obtain a new simplicial complex K_1. Now, the f-vector of K is the sum of the f-vector of K_0 and the f-vector of K_1 shifted in dimension by 2. Namely, $f_k = f_k^0 + f_{k-2}^1$.

In this situation (and for $k>1$) Lemma 3.2 reduces to:

$$\partial^k(m_0+m_1) \leq \max\{\partial^k(m_0), m_1\} + \partial^{k-1}(m_1).$$

By the inductive hypothesis we have $\partial^k(f_{2k+1}^0) \leq f_{2k-1}^0$ and $\partial^{k-1}(f_{2k-1}^1) \leq f_{2k-3}$.

Since by the monotonicity property of simplicial complexes we have $K_1 \subseteq K_0$, we also have $f_{2k-1}^1 \leq f_{2k-1}^0$. Now we easily obtain (for $k>1$):

$$\begin{aligned}
\partial^k(f_{2k+1}) &= \partial^k(f_{2k+1}^0 + f_{2k-1}^1) \\
&\leq \max\{\partial^k(f_{2k+1}^0), f_{2k-1}^1\} + \partial^{k-1}(f_{2k-1}^1) \\
&\leq f_{2k-1}^0 + f_{2k-3}^1 = f_{2k-1}.
\end{aligned}$$

We leave the easy modification in the $k=1$ case to the reader. ∎

Example. If we want to find a $(2d-1)$-dimensional multicomplex with given Betti number β_{2d-1} and with as small f-vector as possible, it is very likely that the minimal f-vector is obtained for the multicomplex of the first β_{2d-1} full square monomials of degree $2d$ in reverse lexicographic order and all of their divisors. For this multicomplex we have (for some $a_d \geq a_{d-1} \geq \ldots \geq a_j \geq j \geq 1$):

$$f_{2d-1} = \beta_{2d-1} = \binom{a_d+d-1}{d} + \binom{a_{d-1}+d-2}{d-1} + \ldots + \binom{a_j+j-1}{j},$$

$$\begin{aligned}
f_{2d-2} = a_d\binom{a_d+d-2}{d-1} &+ \binom{a_{d-1}+d-2}{d-1} + a_{d-1}\binom{a_{d-1}+d-3}{d-2} + \ldots \\
&\ldots + \binom{a_j+j-1}{j} + a_j\binom{a_j+j-2}{j-1},
\end{aligned}$$

$$\beta_{2d-2} = (a_d-1)\binom{a_d+d-2}{d-1}+\binom{a_{d-1}+d-2}{d-1}+(a_{d-1}-1)\binom{a_{d-1}+d-3}{d-2}+\ldots+ \\ +\binom{a_j+j-1}{j}+(a_j-1)\binom{a_j+j-2}{j-1}.$$

For this multicomplex one directly verifies the relation

$$\partial^{d-1}(f_{2d-1}) = f_{2d-2} - \beta_{2d-2}.$$

The reader will easily find all f_k and β_k, but there is no obvious analogue of the above relation.

4. Discussion

Unfortunately, the alternating sum of monomials of codimension 1 dominated (covered in the sense of the partial order) by some fixed monomial is not in general its boundary (as is the case with simplicial complexes). Therefore we cannot detect the homology of a multicomplex directly from its shifted model as for simplicial complexes (see [2]). This is the reason why we cannot use the technique of algebraic shifting. Also, a compressed multicomplex, which as we know minimizes the shadow of the monomials of any given degree, does not maximize the homology of the multicomplex (as for simplicial complexes).

Due to these reasons it is not at all clear that a uniform characterization of (f,β)-pairs can be given for multicomplexes in the same way as for simplicial complexes.

Of course, one source of difficulty might be that the proposed homology theory is not the right one. There are other possible cell complexes fullfilling our general requirements. What is needed is a class of spaces ρ^k, one for each dimension k, such that ρ^k has a CW decomposition with exactly one cell ρ^j in each dimension $j \leq k$. Then we represent each power x^{k+1} by a copy of ρ^k, and general monomials $x_{i_1}^{\alpha_1}\ldots x_{i_k}^{\alpha_k}$ by the join

$$\rho(x_{i_1}^{\alpha_1}\ldots x_{i_k}^{\alpha_k}) = \rho_1^{\alpha_1-1} * \ldots * \rho_k^{\alpha_k-1}.$$

The attaching maps should everywhere be the canonical ones coming from how a cell is attached to ρ^j to get ρ^{j+1}, and their joins. This would give a combinatorially determined CW complex patterned on the given multicomplex. The boundary map of the cellular chain complex, and hence the Betti numbers, would of course be different from those we have used in this paper. One possibility along these lines, that we have not explored since it seems unnatural, is to take real projective k-space with its standard cell decomposition for ρ^k.

Another possible alternative is to take the order homology, i.e. the homology of the simplicial complex of chains of the partially ordered set of monomials (excluding

1) under divisibility. In the case of the square-free monomials defined by a simplicial complex this order complex is the first barycentric subdivision of the original simplicial complex. So, their Betti numbers coincide, and we obtain the natural notion of homology.

In the case of a general multicomplex, however, there is no cell complex with the same f-vector and with the Betti numbers of the associated order complex. In fact, already the Euler-Poincaré relation need not be satisfied for multicomplexes with the order homology. E.g. for $M=\{1,x_1,\ldots,x_n,x_1^2,\ldots,x_n^2\}$ we have

$$(f_0 = f_1 = n,\ \beta_0 = n-1,\ \beta_1 = 0) \Longrightarrow \chi_{-1} = 1-n.$$

In particular, this means that for some multicomplexes with the order homology there is no cell complex having the same f-vector and the same Betti numbers.

We will end by showing that only the square-free part K of a multicomplex M is responsible for its order homology groups. This fact also speaks against using order homology for our purposes. Let us consider the inclusion $i: K \to M$ as a mapping of partially ordered sets. This mapping is of course monotone.

Proposition 4.1. *The order complex of a multicomplex M and the order complex of its subcomplex of square-free monomials have the same homotopy type.*

Proof. We show that the inclusion map i is a homotopy equivalence. According to Quillen's fiber theorem (see [8]) it is enough to prove that the inverse images of all initial segments are contractible.

Let us consider any $m = x_{i_0}^{\alpha_0}\ldots x_{i_k}^{\alpha_k}$. It is easy to verify that $i^{-1}((m]) = (m']$, where $m' = x_{i_0}\ldots x_{i_k}$. Of course, $(m']$ is a cone and therefore contractible. ∎

It follows that the Betti numbers of a multicomplex with the order homology are smaller than or equal to its Betti numbers with the cell complex homology, so the relations from Theorem 3.3 are true here as well.

References

[1] R. N. ANDERSEN, M. M. MARJANOVIĆ, and R. M. SCHORI: Symmetric products and higher-dimensional dunce hats, *Topology Proc.*, **18** (1993), 7–17.

[2] A. BJÖRNER, and G. KALAI: An extended Euler-Poincaré theorem, *Acta Math.*, **161** (1988), 279–303.

[3] A. BJÖRNER, and G. KALAI: Extended Euler-Poincaré relations for cell complexes, in: *Applied Geometry and Discrete Mathematics* (The V. Klee Festschrift) (eds.: P. Gritzmann and B. Sturmfels) Amer. Math. Soc., Providence, R.I., 81–89, 1991.

[4] G. F. CLEMENTS, and B. LINDSTRÖM: A generalization of a combinatorial theorem of Macaulay, *J. Combinatorial Theory*, **7** (1969), 230–238.

[5] J. ECKHOFF, and G. WEGNER: Über einen Satz von Kruskal, *Period. Math. Hungar.*, **6** (1975), 137–142.

[6] C. GREENE, and D. KLEITMAN: Proof techniques in the theory of finite sets, in: *Studies in Combinatorics*, (ed.: G.-C. Rota) Math. Assoc. Amer., Washington, D.C., 22–79, 1978.

[7] G. KALAI: A characterization of f-vectors of families of convex sets in R^d, Part I: Necessity of Eckhoff's conditions, *Israel J. Math.*, **48** (1984), 175–195.

[8] D. QUILLEN: Homotopy properties of the poset of nontrivial p-subgroups of a group, *Advances in Math.*, **28** (1978), 101–128.

[9] R. P. STANLEY: Hilbert functions of graded algebras, *Advances in Math.*, **28** (1978), 57–83.

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